Physics Qualifying Exam UCI May 20, 1974

QUANTUM MECHANICS I

1. a. Using the uncertainty relation, calculate the ground state energy of a hydrogen atom.

   b. Again using the uncertainty relation, calculate the ground state energy of a two electron atom whose nucleus has a charge Ze. Approximate the repulsive electron-electron interaction for electrons at

   $\bar{r}_1$ and $\bar{r}_2$ by $\frac{e^2}{r_1 + r_2}$ since the repulsion most

   likely places them on opposite sides of the nucleus.

2. a. Calculate the scattering amplitude in Born approximation for the gaussian potential

$$V(r) = V_o \, e^{-r^2/2r_o^2}$$

   b. Calculate the total elastic cross section for this potential.

   You may need:

$$\int_{-\infty}^{\infty} dx \, e^{-ax^2+ibx} = \sqrt{\frac{\pi}{a}} \, e^{-\frac{b^2}{4a}}$$

3. Consider two different one dimensional harmonic oscillators perturbed by a third "spring" depending on their displacement.

$$H = \frac{p_1^2}{2m} + \tfrac{1}{2} m\omega_1^2 x_1^2 + \frac{p_2^2}{2m} + \tfrac{1}{2} m\omega_2^2 x_2^2 + K' \, x_1 x_2$$

   a. Find the eigenstates of the unperturbed Hamiltonian, using raising and lowering operators to construct the states.

   b. Using the raising and lowering operator method, evaluate the energies using perturbation theory.

4. The atomic ground state of a muon orbiting about a nucleus is exactly like that of an electron in a single electron atom, with the substitution $m_e \to m_\mu = 106\ \mathrm{MeV}/c^2$ in all formulae. However, because the radius of a muon's orbit is smaller than that of an electron, the finite extent of the nuclear charge distribution will produce a shift of the muon's ground state energy from its value for a point nucleus. Calculate this energy shift to first order in perturbation theory for a muon bound to an iron nucleus ($Z = 26$, radius $R \approx 4 \times 10^{-13}$ cm). Assume the iron nucleus to be a uniformly charged sphere, for which the electrostatic potential is

$$\varphi(r) = \frac{Ze}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] \qquad r \le R$$

5. The muon is a spin $\frac{1}{2}$ particle of mass $106\ \mathrm{MeV}/c^2$, with the magnetic moment of a point Dirac particle ($g = 2$). Consider a negative muon captured in an atomic state by a nucleus. The increased mass of the muon relative to an electron will have an effect on all aspects of such a state. Make a quantitative comparison of the muon's atomic states with the corresponding electronic states, for each of the following aspects. In each case, give a physical explanation for the effect of the muon's great mass on that aspect (based on a Bohr model of the atom and your understanding of the relationship of magnetic moments, angular momenta, etc.). Note that the radius of a muon's orbit is smaller than that of an electron by a factor (m_e/m_μ).

   a. principle level structure
   b. fine structure (strength of spin-orbit coupling)
   c. hyperfine structure
   d. Zeeman splitting of levels in an external field.

6. A spin $\frac{1}{2}$ particle decays at rest into a spin $\frac{1}{2}$ and a spin 0 particle. The final state is a superposition of orbital angular momentum 0 and 1. (Parity is not conserved in the decay). The decay amplitudes leading to these orbital angular momentum states are a_s and a_p. Derive the expression for tne angular distribution of the decay products in terms of a_s, a_p, and the polarization of the initial particle. Use the Clebsch-Gordan coefficients $C(J, M, j_1, m_1, j_2, m_2)$ and the spherical harmonics Y_ℓ^m :

$$Y_o^o = \sqrt{1/4\pi} \qquad C(\tfrac{1}{2},\tfrac{1}{2},1,1,\tfrac{1}{2},-\tfrac{1}{2}) = \sqrt{2/3}$$

$$Y_1^o = \sqrt{3/4\pi}\cos\theta \qquad C(\tfrac{1}{2},\tfrac{1}{2},1,0,\tfrac{1}{2},\tfrac{1}{2}) = \sqrt{1/3}$$

$$Y_1^{\pm 1} = \mp\sqrt{3/8\pi}\sin\theta\, e^{\pm i\phi}$$

Quantum Mechanics II UCI May 21, 1974

1. a. Find the complete wave function for the ground state of two electrons in a one dimensional box.

   b. Evaluate the energy due to a weak spin-spin interaction $V_o \bar{S}_1 \cdot \bar{S}_2$.

   c. Write down the lowest energy wave functions for three electrons in the one dimensional box.

2. a. Consider two 3 dimensional potential wells: a finite "square" well $(V = -V_o \quad r < R$

$$V = 0 \quad r > R$$

   and an infinite square well $(V = -V_o \quad r < R$

$$V = +\infty \quad r > R \quad .$$

   In which well will the energies of the first bound state be higher? Why?

   b. The $\ell = 0$ energy levels of a particle bound in a potential well have separations which increase with energy for an infinite square well potential, are constant for a harmonic oscillator potential, and decrease with energy for a Coulomb potential. Explain this behavior qualitatively in terms of the shapes of these three potentials.

3. The deuteron is known to have $J = 1$ and to be a bound state of a neutron and a proton which is predominantly $L = 0$ $S = 1$ It has a small positive quadrupole moment, implying that it is slightly cigar shaped, and not pure $L = 0$ $S = 1$.

   a. What combinations of L and S might form a $J = 1$ state?

   b. Of these, which will not be expected to mix with the $L = 0$ $S = 1$ state in the deuteron? Why?

c. The mixing of states with different L implies a noncentral np potential which might depend linearly on various combinations of L, S, and the relative positions and momenta of the nucleons, r and p. Which of the following dependances will not mix values of L <u>or</u> must otherwise be excluded? Give reasons.

$r \cdot S$ $(L \cdot S)^2$

$(r \cdot S)^2$ $p \cdot S$

$L \cdot S$

d. Classically a spinning object will tend to bulge at the equator. How can a deuteron be cigar shaped?

4. Consider a free atom.

a. What are the selection rules on ℓ and m_ℓ for electric dipole radiation?

b. Why is one angular momentum coupling ruled out for dipole radiation?

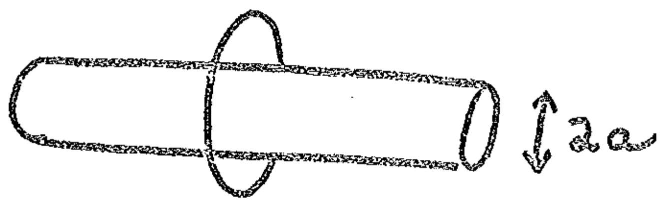
c. What transitions are strictly forbidden?

d. Roughly by what factor is the $n = 3$, $\ell = 2$ decay directly to the ground state suppressed with respect to the $n = 3$, $\ell = 1$ decay to the ground state of a hydrogen atom.

Physics Qualifying Exam UCI May 22, 1974

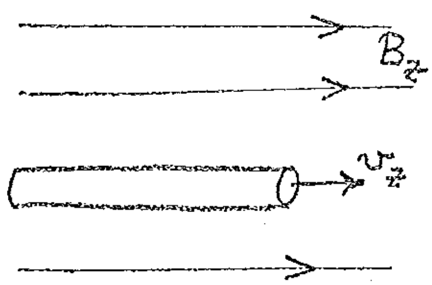
ELECTRICTY AND MAGNETISM

1. An infinite solenoid has n turns of wire per unit length and radius a. Outside this solenoid is a thin coil, of radius b, concentric with the solenoid and having N turns of wire in it. Find the mutual inductance between these two objects.

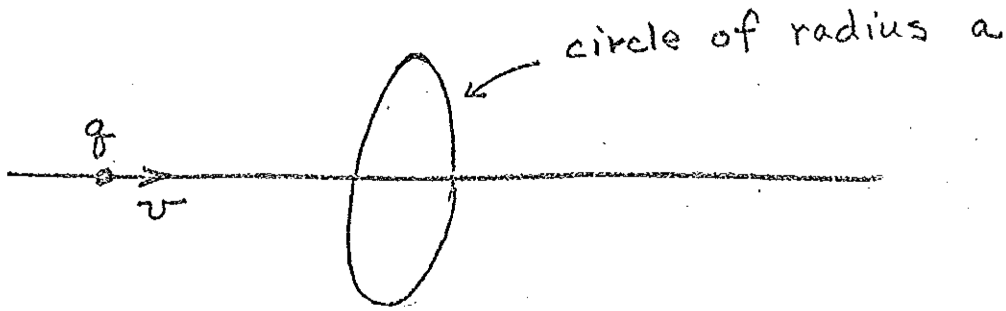


2. High energy X-rays (of frequency ω) propagate in vacuum until they strike a metal plate. The electrons inside are free (not bound to individual ions) and have density n. Find the critical angle of incidence for which the X-rays will be completely reflected by the plate.

3. An infinitely long cylindrical current distribution, uniform to radius a, produces its own self-field B_θ as it propagates along an imposed homogeneous field B_z. The electrons have a net charge density per unit length Q. The electrons have some perpendicular motion, $v_\perp \ll v_z$. Also, $B_\theta \ll B_z$. To first order in $v_\perp/v_z$ and B_θ/B_z, find the frequencies of the electron motion. What sort of orbits do these frequencies represent?



4. A charge q moves with uniform velocity v (where v is very much less than c) along the axis of a circle C of radius a:



a. Calculate the magnetic field at C as a function of the time, t, using Ampère's law. Get $\oint \underline{B} \cdot d\underline{x}$ around C.

b. Let ψ = electric flux through C = $\int_S \underline{E} \cdot \underline{n}\, da$, where S = circular disc bounded by C. Calculate ψ as a function of t using Coulomb's law.

c. Verify $\frac{\partial \psi}{\partial t} = c^2 \oint \underline{B} \cdot d\underline{x}$.

(Note: Constant factor will depend on the units. M.k.s. units are used above.)

d. How does $\psi(t)$ change just as q crosses center of C? Explain why $\oint \underline{B} \cdot d\underline{x}$ is continuous anyway.

5. The most commonly accepted model of a pulsar assumes it is rotating at angular frequency Ω and is an excellent conductor inside, so

$$\underset{\sim}{E} \times \frac{v}{c} \times \underset{\sim}{B} = 0$$

inside. $\underset{\sim}{B}$ is constant and normal to the equatorial plane everywhere inside. Here $\underset{\sim}{v} = \underset{\sim}{\Omega} \times \underset{\sim}{r}$. Suppose outside this rotating sphere of radius R there is a vacuum.

a. Find the electrostatic potential for $r > R$.

b. From this, find the surface charge density.

c. Find the electric field at the surface. How does the electrical force compare with the gravitational force on an electron?

6. Consider a spherical planet (radius R) made of insulator. It rotates with angular frequency Ω and has gravitational acceleration g'. At an angle θ measured from the pole we place, infinitesimally above the planet, a small disk made of insulator, having mass m. Then we begin introducing equal charge densities σ uniformly on the surfaces of both the planet and the disk. At what value of charge density σ will the disk begin to rise off the planet? (Assume the disk rotates with the planet.)

Mathematical Physics

1. Find the Fourier transform of $\psi_n(x)$ given that

$$\psi_n''(x) + (2n + 1 - x^2)\psi_n(x) = 0$$

$$\psi_n(x) = 2^{-n/2}\,\pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}}e^{-x^2/2}\,H_n(x)$$

$$\int_{-\infty}^{\infty} \psi_n^2(x)\,dx = 1$$

$$\psi_n(-x) = (-1)^n\psi_n(x)$$

2. Evaluate the integral

$$\int_0^{\infty} \frac{\sin ay\,dy}{e^{2y} - 1}$$

3. Assume that f(x) is a well-behaved function and $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists. Prove that if $F(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{f(x')\,dx'}{x' - x}$, then

$$f(x) = \frac{-P}{\pi}\int_{-\infty}^{\infty} \frac{F(x')\,dx'}{x' - x}$$

(P stands for the principal part of the integral.)

4. Give examples of hyperbolic, elliptic and parabolic linear differential equations. Discuss what boundary conditions result in a unique stable solution in each case.

5. Prove that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} dx' \left\{\frac{2}{\pi} \frac{\sin^2[k(x-x')/2]}{k(x - x')^2}\right\} f(x') = f(x)$$

$$\text{or } \lim \frac{2}{\pi} \frac{\sin^2 kx/2}{kx^2} = \delta(x)$$

6. Take the potential function

$$V(x) = \infty \qquad x < 0$$

$$= ax \qquad x > 0$$

In the one-dimensional Schrodinger equation,

$$\left(- \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) - E \right) \psi(x) = 0$$

you are to use one of the following trial functions to estimate the energy of the lowest bound state.

(1) Of the two choices,

(a) $\psi(x) = 2\beta^{3/2} x e^{-\beta x}$, $\quad x > 0$

$\psi(x) = 0$, $\quad x < 0$

(b) $\psi(x) = A x^2 e^{-c^2 x^2}$ $\quad$ (all x)

which do you think is the best initial choice?

5. (Continued)

(2) Find an approximate value of the ground state energy, using your choice from part (1). (Assume $\psi(x)$ is already normalized to unity.)

7. Consider a molecule which is in the shape of a tetrahedron (and has "T" as its symmetry group). Every electron in this molecule must of course also have this same symmetry, i.e., all electron wave functions must be basis functions for some irreducible representation of the group T. Suppose we apply a uniform magnetic field (Zeeman effect) parallel to one of the 3-fold symmetry axes. Show how each molecular electronic level splits.

$$\omega = 2\pi i/3, \quad \varepsilon = 2\pi i/6.$$

23 T	E	C_{2m}	C_{3j}^+	C_{3j}^-
A $x^2+y^2+z^2$	1	1	1	1
E $(x^2-y^2, (2z^2-x^2-y^2)/\sqrt{3})$	1	1	ω	ω^*
	1	1	ω^*	ω
T $(x, y, z), (I_x, I_y, I_z), (yz, zx, xy)$	3	-1	0	0

3 C_3	E	C_3^+	C_3^-
A z, I_z	1	1	1
E $(x, y), (I_x, I_y)$	1	ω	ω^*
	1	ω^*	ω

$3m$ C_{3v}	E	$C_3^\pm$	σ_{di}
A_1 x^2+y^2, z, z^2	1	1	1
A_2 I_z	1	1	-1
E $(x, y), (zx, zy), (I_x, I_y), \left(\frac{x^2-y^2}{\sqrt{2}}, xy\right)$	2	-1	0

$\bar{6}$ C_{3h}	E	S_3^-	C_3^+	σ_h	C_3^-	S_3^+
A' I_z	1	1	1	1	1	1
A'' z	1	-1	1	-1	1	-1
E'' (I_x, I_y)	1	ε	ε^2	-1	ε^4	ε^5
	1	ε^5	ε^4	-1	ε^2	ε
E' (x, y)	1	ε^2	ε^4	1	ε^2	ε^4
	1	ε^4	ε^2	1	ε^4	ε^2

I. Consider a system of N_A particles of type A and N_B of type B. Particles B are <u>fixed</u> at lattice sites R_j; j=1, ... N_B and those of type A are free to move. A force between an A particle at position r and a B particle at position R is due to a potential

$$v(\underset{\sim}{r}-\underset{\sim}{R})$$

whose range is less than one half the lattice separation between B particles. Find the equation of state for this system. Express your answer in terms of

$$f = \int \left[e^{-\beta v(\vec{r}-\vec{R})} - 1 \right] d\vec{r} \quad .$$

II. A substance has the following properties.

(i) At a constant temperature T_o the work done by it on expansion from V_o to V is

$$W = RT_o \ln \quad (V/V_o) \, .$$

(ii) The entropy is given by

$$S = R \frac{V_o}{V} \left(e^{T/T_o} - 1\right)$$

where V_o, T_o are fixed constants.

(a) Calculate the Helmholtz free energy

(b) Find the equation of state

(c) Find the work done at an arbitrary temperature T.

III. a. Obtain the pressure and density of an ideal Bose-Einstein gas in <u>two</u> dimensions. Express your answer in terms of the functions

$$g_n(z) = \frac{1}{\Gamma(n)} \int \frac{x^{n-1}dx}{z^{-1}e^x-1}$$

b. Does this system exhibit a Bose-Einstein condensation?

IV. Obtain the relation between the pressure and internal energy of an ultrarelativistic Boltzmann gas in d dimensions. Discuss the result.

V. Consider a system with a gap in its energy spectrum. Specifically let the ground state have $E_o=0$ and the next state's start at $E=\Delta$ where Δ is non-zero for all N and V. Show that the specific heat C_v goes to zero as

$$C_v \sim \text{const. } e^{-(\Delta/kT)}$$

as $T \to 0$. Use either Boltzmann or Bose statistics.

VI. Consider a system of N noninteracting particles each having two internal energy levels E_o and E_1. Find the specific heat at constant volume of this system. (Use Classical Stat. Mechanics).

Physics Qualifying Exam UCI May 20, 1974

(4 points) 1. A man standing on the edge of a cliff at some height above the ground below throws one ball straight up with initial speed u and then throws another ball straight down with the same initial speed. Which ball, if either, has the larger speed when it hits the ground? Neglect air resistnace.

(4) 2. A solid wooden sphere rolls down two different inclined planes of the same height but different inclines. (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one incline than the other? If so, which one and why?

(4) 3. Under what sets of conditions is $\bar{N} = \frac{d\bar{L}}{dt}$ valid? ($\bar{N}$ is the total torque on a system, and $\bar{L}$ is its total angular momentum.)

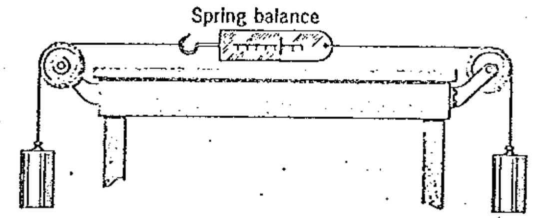
(4) 4. Define "virtual displacement," as used in the Lagrangian formulation of classical mechanics.

(4) 5. Why do the (holonomic) forces of constraint not appear explicitly in the Lagrangian equations of motion?

(4) 6. A satellite is in uniform circular motion about the earth. How much work is done on it during each revolution if its mass is m, its speed is v, and the radius of its orbit is R?

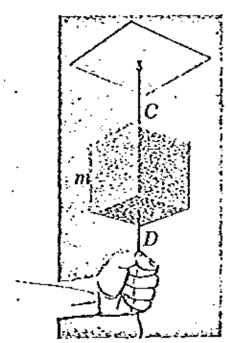
(4) 7. Consider a one-dimensional elastic collision between a given incoming body A and a body B initially at rest. How would you choose the mass of B, in comparison to the mass of A, in order that B should recoil with (a) the greatest momentum, (b) the greatest kinetic energy, and (c) the greatest speed?

8. (4) Two 5 lb weights are attached to a spring scale as shown in the Figure. Does the scale read 0 lb, 5 lb, 10 lb, or give some other reading?



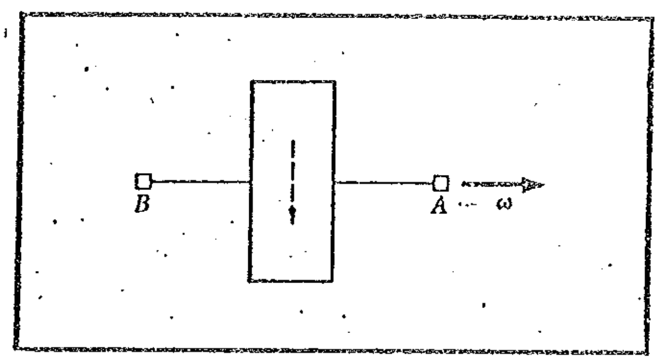
9. (4) Can a sailboat be propelled by air blown at the sails from a fan attached to the boat?

10. (4) A block of mass m is supported by a cord C from the ceiling, and another cord D is attached to the bottom of the block. Explain this: If you give a sudden jerk to D it will break, but if you pull on D steadily, C will break.



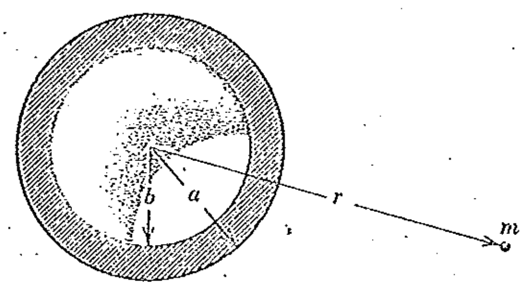
11. A ball is rolling (without slipping) on a horizontal plane.
(4) It then rolls (without slipping) up an inclined plane. (There must of course be friction present in order for it to roll without slipping.) Now suppose that the above inclined plane is replaced by an identical frictionless one (horizontal plane remains unchanged). Compared to the plane with friction, will the ball travel a greater, lesser, or the same distance up the frictionless plane before momentarily stopping and beginning to go back down the plane? Why? (Assume the initial state is the same in both cases, i.e. it is rolling without slipping on the horizontal plane.)

12. The figure represents a gyroscope wheel seen from one side,
(4) with its axis mounted in bearings A and B. It is spinning with angular velocity as shown, the near side of the wheel moving downward. Upward support forces exist equally at A and B. It is now desired to reorient the wheel to place A directly over B, without moving the center of mass of the system. Describe the additional forces to be applied at A and B.

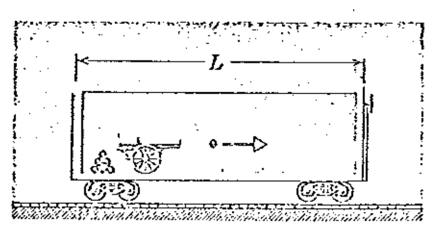


13. A ball of radius r and mass m rolls without slipping on
(6) the inside of a large hemispherical bowl of radius R. If the ball starts from rest at the top edge of the bowl, calculate the force that the bowl exerts on the ball at the bottom of the bowl.

14. A sphere of matter, radius a, has a concentric cavity,
(7) radius b, as shown in cross section in the figure. (a) Sketch the gravitational force F exerted by the sphere on a particle of mass m, located a distance from the center of the sphere, as a function of r in the range $0 \le r \le \infty$. Consider points $r = 0, b, a,$ and ∞ in particular. (b) Sketch the corresponding curve for the potential energy V(r) of the system.



15. A cannon and a supply of cannon balls are inside a sealed,
(7) impenetrable railroad car as in the Figure. The cannon fires to the right, the car recoiling to the left. The cannon balls remain in the car after hitting the far wall. Show that no matter how the cannon balls are fired the railroad car cannot travel more than its length L, assuming it starts from rest.



16. An hour glass is weighted on a sensitive balance, first when
(7) sand is dropping in a steady stream from the upper to the lower part and then again after the upper part is empty. Are the two weights the same? Explain.

17. Consider a uniform sphere of soft rubber (or jello).
(7) (a) Write down its moment of inertia tensor (about its geometrical center) in matrix form, i.e.

$$I = k \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

where k is some overall constant which you need not determine.
(b) Suppose this sphere now rotates about a diameter (call it the z-axis). (It of course then changes its shape due to the "centrifugal" force. However, assume its density remains constant.) Write down its new moment of inertia (about its geometrical center). Clearly indicate how each element is related to every other element in the matrix. Also clearly indicate how each element is related to its corresponding one in part (a) above.

In each of the following problems:

(a) Find the Lagrangian
(b) Find the Hamiltonian
(c) List all holonomic and nonholonomic constraints
(d) State whether the external (i.e. nonconstraint) forces are conservative or not
(e) List the ignorable coordinates
(f) State whether or not H = T+V. Whether $\frac{dH}{dt} = 0$ or not.

18. A gas molecule moving in a cubical box with gravity acting.
(6)

(6) 19. A particle moving in the attractive central field with

$$\bar{F} = -(k/r^2)\, e^{-\alpha t}\, \hat{r} .$$

(6) 20. A bead sliding on a rough wire bent in the form of a helix (cylindrical coordinates: $z=a\theta$, $\rho=b$, a and b are constants and ρ, θ, z are the usual cylindrical coordinates.) The origin is a center of attractive force $\bar{F} = -kr^3\hat{r}$ where r is the distance from the bead to the origin.